1

LECTURE 15: last time — weak Markov property + construction.

Thm: (Stronger Markov property) (Y_n) MC on S, $f: S^{\mathbb{N}} \to \mathbb{R}$ is bounded + measurable. Then,

$$\mathbb{E}_{\mu_0}\left[f(Y_n, Y_{n+1}, \dots) \mid \mathcal{F}_n\right] = \mathbb{E}_{Y_n}\left[f(Y_0', Y_1', \dots)\right]$$

Pf Sketch: Another hierarchy of approximations for f:

(1) $f = \mathbb{1}\{Y_{n+1} \in A\}$ $\longrightarrow$ by weak Markov

(2) $f = \mathbb{1}\{Y_n \in A_0, Y_{n+1} \in A_1, \dots, Y_{n+m} \in A_m\}$ $\rightsquigarrow$ integral formula + induction.

(3) General: approx by simple fn. + monotone class thm.

Def: $p^{(1)}(y, A) := p(y, A) = \mathbb{P}_y[Y_1 \in A]$.

$$p^{(k)}(y, A) := \mathbb{P}_y[Y_k \in A] = \int_S dp_y(y_1) \int_S dp_{y_1}(y_2) \cdots \int_A dp_{y_{k-1}}(y_k)$$

Exc: $p_y^{(k)}(A) := p^{(k)}(y, A)$ is a prob. measure, and in finite/countable case, $p^{(k)}(x, \{y\}) := M^{(k)} \rightsquigarrow M^{(k)} = (M^{(1)})^k$.

Cor: (Chapman-Kolmogorov eq.) $p^{(m+n)}(y, A) = \mathbb{P}_y[Y_{m+n} \in A]$

$$= \int_{S \ni z} \mathbb{P}_z[Y_n \in A]\, dp_y^{(m)}(z) \qquad \left(\text{countable: } \sum_{z \in S} p^{(m)}(y, z)\, p^{(n)}(z, A)\right).$$

Pf: $\mathbb{P}_y[Y_{m+n} \in A] = \mathbb{E}_y[\mathbb{1}_A(Y_{m+n})] = \mathbb{E}\Big[\underbrace{\mathbb{E}[\mathbb{1}_A(Y_{m+n}) \mid \mathcal{F}_m]}\Big]$

$$= \mathbb{E}_y\left[\mathbb{E}_{Y_m}[\mathbb{1}_A(Y_n')]\right] = \mathbb{E}_y\, \mathbb{P}_{Y_m}[Y_n' \in A].$$

Thm: (Strong Markov property) N stopping time. Recall $\mathcal{F}_N = \{A : \forall n,\ A \cap \{N \le n\} \in \mathcal{F}_n\}$.

For $f : S^{\mathbb{N}} \to \mathbb{R}$ bdd, on the event $\{N < \infty\}$,

$$\mathbb{E}_{\mu_0}\left[f(Y_N, Y_{N+1}, \dots) \mid \mathcal{F}_N\right] = \mathbb{E}_{Y_N}\left[f(Y_0', Y_1', \dots)\right].$$

Pf: Need to check, $\forall A \in \mathcal{F}_N$, $A \subseteq \{N < \infty\}$,

$$\mathbb{E}_{\mu_0} \mathbb{1}\{A\} f(Y_N, Y_{N+1}, \dots) \overset{(?)}{=} \mathbb{E}_{\mu_0} \mathbb{1}\{A\}\, \mathbb{E}_{Y_N}\left[f(Y_0', Y_1', \dots)\right]$$

$$\downarrow$$

$$= \mathbb{E}_{\mu_0} \sum_{n=0}^{\infty} \mathbb{1}\{\underbrace{A, N=n}_{\in \mathcal{F}_n}\} f(Y_n, Y_{n+1}, \dots) = \sum_{n=0}^{\infty} \mathbb{E}_{\mu_0} \mathbb{1}\{A, N=n\}\, \mathbb{E}_{Y_n} f(Y_0', Y_1', \dots).$$

S countable: $p^{(k)}(x,y) := p^{(k)}(x, \{y\})$.

Def: $T_s^{(0)} := 0$, $T_s^{(k+1)} := \min\{n > T_s^{(k)} : Y_n = s\}$ (time of $(k+1)^{\text{th}}$ visit to s).

$\rho_{xy} := \mathbb{P}_x[T_y^{(1)} < \infty] = \mathbb{P}[\text{reach } y \text{ from } x]$.

$(0 = T_s^{(0)} < T_s^{(1)} < T_s^{(2)} < \dots)$

Thm: $\mathbb{P}_x[T_y^{(k)} < \infty] = \rho_{xy}\rho_{yy}^{k-1}$

Def: $x \in S$ recurrent if $\rho_{xx} = 1$, otherwise transient.

Cor: (1) x recurrent $\Rightarrow \mathbb{P}_x[\text{return infinitely many times}] = 1$.

(2) y recurrent $\Rightarrow \mathbb{P}_x[\text{visit } y \text{ infinitely many times}] = \rho_{xy}$. $\forall x$

(3) y transient $\Rightarrow \mathbb{P}_x[\text{ — " — }] = 0$. $\forall x$.

(3)

<u>Pf of Thm</u>: $k=1$. ✓ $k \geq 2$: $f(Y_0, Y_1, \dots) := \mathbb{1}\{\text{any } Y_i = y\}$ for $i \geq 1$.

On $\{T_y^{(k-1)} < \infty\}$,

$$\mathbb{E}_x\left[\underbrace{f\left(Y_{T_y^{(k-1)}}, Y_{T_y^{(k-1)}+1}, \dots\right)}_{= \mathbb{1}\{T_y^{(k)} < \infty\}} \,\middle|\, \mathcal{F}_{T_y^{(k-1)}}\right] = \mathbb{E}_{Y_{T_y^{(k-1)}}} f(Y_0', Y_1', \dots)$$

$$= \mathbb{E}_y f(Y_0', Y_1', \dots)$$

$$\Longrightarrow \mathbb{P}_x\left[T_y^{(k)} < \infty \,\middle|\, F_{T_y^{(k-1)}}\right] \qquad = \mathbb{E}_y \mathbb{1}\{\text{ever return}\} = \rho_{yy}.$$

$$= \rho_{yy}\, \mathbb{1}\{T_y^{(k-1)} < \infty\} \Longrightarrow \mathbb{P}_x\left[T_y^{(k)} < \infty\right] = \rho_{yy}\, \mathbb{P}_x\left[T_y^{(k-1)} < \infty\right].$$

<u>Def</u>: $N(y) := \#\{\text{visits to } y\} = \sum_{n=1}^{\infty} \mathbb{1}\{Y_n = y\}$.

<u>Thm</u>: y recurrent $\Longleftrightarrow$ $\mathbb{E}_y N(y) = \infty$.

<u>Pf</u>: $\mathbb{E}_y N(y) = \sum_{k=1}^{\infty} \mathbb{P}_y[N(y) \geq k] = \sum_{k=1}^{\infty} \mathbb{P}_y[T_y^{(k)} < \infty] = \sum_{k=1}^{\infty} \rho_{yy}^k$.

<u>Rk</u>: $\mathbb{E} N(y) = \sum_{n=1}^{\infty} \mathbb{P}[Y_n = y] \rightsquigarrow$ Borel-Cantelli: "tight" here.

<u>Application</u>: (Y_n) = SRW on $\mathbb{Z}^d$. <u>Q</u>: Is $y = 0$ transient? [Polya].

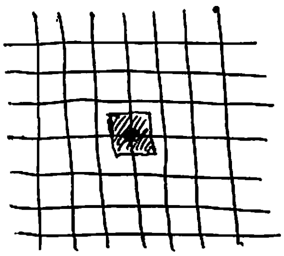
$\mathbb{E} N(y) = \sum_{n=1}^{\infty} \mathbb{P}[Y_n = 0]$. $\quad e_i := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$. Each step: $\pm e_i$

$$Y_n = \sum_{i=1}^{n} v_i \quad \text{for } v_i \overset{iid}{\sim} \mathrm{Unif}(e_1, -e_1, e_2, -e_2, \dots, e_d, -e_d).$$

Expect: CLT should apply to Y_n.

$\mathbb{E} v_i = 0$, $\mathrm{Cov}(v_i) = \mathbb{E} v_i v_i^T = \frac{1}{d} I_d$.

$\rightarrow \frac{1}{\sqrt{n}} Y_n \approx \mathcal{N}(0, \frac{1}{d} I_d)$, or $Y_n \approx \mathcal{N}(0, \frac{n}{d} I_d)$.



$$\mathbb{P}[Y_n = 0] = \mathbb{P}\left[Y_n \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^d\right]$$

$$\approx \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \frac{1}{\sqrt{\det(2\pi \cdot \frac{n}{d} I_d)}} \exp\Big(\underbrace{-\frac{1}{2} z^T \left(\tfrac{n}{d} I_d\right)^{-1} z}_{\approx 0}\Big)\, dz$$

$$\approx \frac{c_d}{n^{d/2}} \quad \text{(for large } n\text{)}$$

$$\Longrightarrow \mathbb{E} N(y) \approx c_d \sum_{n=1}^{\infty} \frac{1}{n^{d/2}}$$

Thm: (Polya) 0 is recurrent if $d = 1, 2$; transient if $d \geq 3$.