(1)

LECTURE 14: Markov chains over general state space S.

$(Y_n)_{n\geq 0}$ adapted to $(\mathcal{F}_n)_{n \geq 0}$, Y_n r.v. in S, i.e. $Y_n : \Omega \longrightarrow S$.

Need $(S, \mathcal{S})$ measurable space, Y_n an $\mathcal{F}_n$-m'able function.

Want: $\mathbb{P}[Y_{n+1} \in A \mid \mathcal{F}_n] = \mathbb{P}[Y_{n+1} \in A \mid Y_n] = p(Y_n, A)$

$$\mathbb{P}[Y_{n+1} \in A \mid \mathcal{F}_n] = \mathbb{E}[\mathbb{1}_A(Y_{n+1}) \mid \mathcal{F}_n], \qquad \mathbb{P}[Y_{n+1} \in A \mid Y_n] = \mathbb{E}[\mathbb{1}_A(Y_{n+1}) \mid Y_n], \qquad p : S \times \mathcal{S} \longrightarrow [0,1].$$

Idea: "p = law of $Y_{n+1} \mid Y_n$". Rk: For fixed A, must be m'able fn. of Y_n.

Def: For X, Y r.v. on $(\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{S})$, $p : S \times \mathcal{S} \longrightarrow [0,1]$ is a regular conditional probability of Y on X if:

(1) $\forall A \in \mathcal{S}$, $\mathbb{P}[Y \in A \mid X] = p(X, A)$ a.s.

(2) $\forall A \in \mathcal{S}$, $x \longmapsto p(x, A)$ is m'able function of $x \in S$.

(3) $\forall x \in S$, $A \longmapsto p(x, A)$ is prob. measure.

Rk: Does not always exist! But ok if $(S, \mathcal{S})$ nice, e.g. countable or $\mathbb{R}^n$.

(2) + (3) $\longrightarrow$ p valid Markov/transition kernel.

Def: (Y_n) is Markov chain w/r/t $(\mathcal{F}_n)$ and w/ kernel p if:

(1) Adapted to $(\mathcal{F}_n)$

(2) $\forall n \geq 0$, $A \in \mathcal{S}$, $\underbrace{\mathbb{P}[Y_{n+1} \in A \mid \mathcal{F}_n] = p(Y_n, A)}$ a.s.

$$\left.\begin{aligned} \text{"}p(Y_n, \cdot) &= \mathrm{Law}(Y_{n+1} \mid Y_n) \\ &= \mathrm{Law}(Y_{n+1} \mid \mathcal{F}_n)\text{"} \end{aligned}\right\} \text{MARKOV PROPERTY.}$$

Q: Does MC specified by p exist? Unique? A: Need starting point/measure.

②

Given μ_0 on $\mathcal{S}$ and p, can <u>generate</u> MC: $p_y(A) := p(y, A)$

<u>Intuitively</u>: $Y_0 \sim \mu_0$, $Y_1 \sim p_{Y_0}$, $Y_2 \sim p_{Y_1}$, ... $\leftarrow$ <u>sampling algorithm</u>

Corresponding mathematical objects: define μ_n on $\mathcal{S}^{n+1}$ = marginal of $(Y_0, \dots, Y_n)$

by $\mu_n(A_0 \times A_1 \times \dots \times A_n)$ (" $= \mathbb{P}[Y_0 \in A_0, Y_1 \in A_1, \dots, Y_n \in A_n]$ ")

$$:= \int_{\substack{A_0 \\ \ni \\ y_0}} d\mu(y_0) \int_{\substack{A_1 \\ \ni \\ y_1}} dp_{y_0}(y_1) \int_{\substack{A_2 \\ \ni \\ y_2}} dp_{y_1}(y_2) \cdots \int_{\substack{A_n \\ \ni \\ y_n}} dp_{y_{n-1}}(y_n).$$

<u>Ex</u>: $\mu_1(A_0 \times A_1) = \int_{A_0} d\mu(y_0) \int_{A_1} dp_{y_0}(y_1) = \int_{A_0} d\mu_{y_0}\, p(y_0, A_1)$.

Then: extend μ_n to measure on $\mathcal{S}^{n+1}$, check compatibility $\longrightarrow$ <u>Kolmogorov extension</u>.

$\rightsquigarrow \mu_\infty =: \mathbb{P}_{\mu_0}$ on $(S^{\mathbb{N}}, \mathcal{S}^{\mathbb{N}})$.

$\hookrightarrow$ = law of MC w/ transition kernel p, started @ μ_0.

$\mathbb{P}_y := \mathbb{P}_{\delta_y}$.

<u>Prop</u>: $\mathbb{P}_{\mu_0}[\underset{\substack{\uparrow \\ \ni \\ \mathcal{S}^{\mathbb{N}}}}{A}] = \int \mathbb{P}_y[A]\, d\mu_0(y)$. (Exercise.)

<u>Thm</u>: This (Y_n) is MC w/r/t $\mathcal{F}_n = \sigma(Y_0, Y_1, \dots, Y_n)$, kernel p.

I.e., $\mathbb{P}_{\mu_0}[Y_{n+1} \in A \mid \mathcal{F}_n] = p(Y_n, A)$ a.s. (i.e., construction "worked")

<u>Pf Sketch</u>: $\mathbb{E}[\mathbb{1}_A(Y_{n+1}) \mid \mathcal{F}_n]$. Need to check: $\forall B \in \mathcal{F}_n$ (i.e., B only "about" $Y_0, \dots, Y_n$),

$$\mathbb{E}_{\mu_0}[\mathbb{1}\{Y_{n+1} \in A, B\}] \overset{(?)}{=} \mathbb{E}_{\mu_0}[p(Y_n, A)\mathbb{1}\{B\}]$$

First: $B = \mathbb{1}\{Y_0 \in A_0, \dots, Y_n \in A_n\}$

"

$$\int_{A_0} d\mu(y_0) \cdots \int_{A_n} dp_{y_{n-1}}(y_n) \underbrace{\int_A dp_{y_n}(y_{n+1})}_{p(y_n, A)}$$

① Approximation argument for $p(y_n, A)$.

② Extend to general B by π-λ Thm.