①

LECTURE 13: Branching process $(Z_n)_{n \geq 0}$ w/ child variables $(X_{n,i} \overset{\text{iid}}{\sim} \mu)$, $m = \mathbb{E} X_{n,i} > 1$, $p_k := \mathbb{P}[X_{n,i} = k]$.

$$\rho := \mathbb{P}[\underbrace{Z_n = 0 \ \forall n \text{ suff. large ("extinction")}}_{\text{event } E}]$$

<u>Thm</u>: (last time) $\rho < 1$.

<u>Q</u>: Is $Z_n \approx m^n = \mathbb{E} Z_n$ when E does not happen?

$$M_n := \frac{Z_n}{m^n} \text{ mgl}, \quad \mathbb{E} M_n = 1, \quad M_n \xrightarrow{\text{a.s.}} M_\infty.$$

<u>Thm</u>: If $\mathbb{E} \overset{\swarrow m'}{X_{1,1}^2} < \infty$, then $M_n \xrightarrow{\text{a.s.}, L^2, L^1} M_\infty$. In particular, $\mathbb{E} M_\infty = 1$, $\exists\, \delta, \varepsilon > 0$ s.t. $\underbrace{\mathbb{P}[M_n \geq \delta]}_{= \mathbb{P}[Z_n \geq \delta m^n]} \geq \varepsilon$, $\forall n$ suff. large.

<u>Pf</u>: Use L^2 MC. Need to bound:

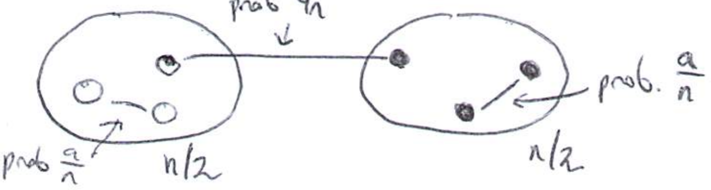
$$\mathbb{E} M_n^2 = \frac{1}{m^{2n}} \mathbb{E} Z_n^2 = \frac{1}{m^{2n}} \mathbb{E}\Big[\mathbb{E}\big[\overset{\uparrow\, \left(\sum_{i=1}^{Z_{n-1}} X_{n,i}\right)^2}{Z_n^2} \,\big|\, \mathcal{F}_{n-1}\big]\Big]$$

$$\overset{\text{(Exc.)}}{=} \frac{1}{m^{2n}} \mathbb{E}\left[Z_{n-1} m' + Z_{n-1}(Z_{n-1} - 1) m^2\right]$$

$$= \frac{m' - m^2}{m^{n+1}} + \mathbb{E}[M_{n-1}^2] \leq (m' - m^2) \sum_{n=0}^{\infty} \frac{1}{m^{n+1}} < \infty.$$

<u>Thm</u>: (Kesten-Stigum '66) M_∞ not a.s. $= 0 \iff \mathbb{E} X_{11} \log(1 + X_{11}) < \infty$.

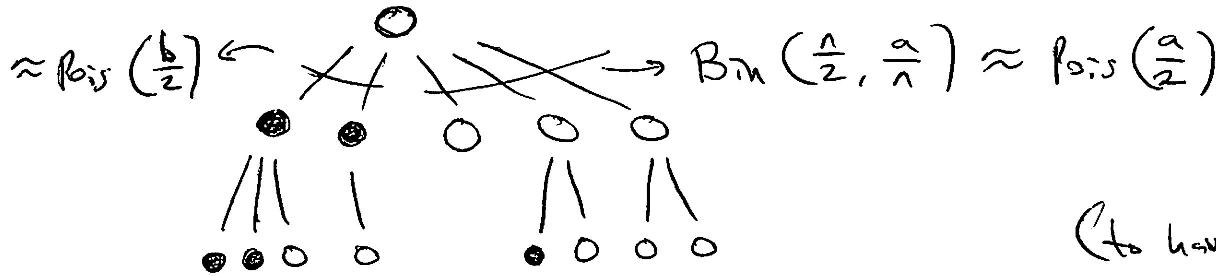
<u>Stochastic Block Model</u>: random graph on $\{1, \ldots, n\}$



degree $\sim \mathrm{Bin}\left(n, \frac{a+b}{2n}\right) \approx \mathrm{Pois}\left(\frac{a+b}{2}\right)$

$\hookrightarrow$ sparse random graph $\approx$ social network.

② Locally around a vertex $\approx$ <u>multi-type / colored branching process</u>:



(to have concentration.)

Alg. idea to recover partition: iterative majority vote at large depth.

<u>Kesten-Stigum '66 (II)</u>: Limit theory, mean $m \to$ $M = \begin{array}{c|cc} & \circ & \bullet \\ \circ & a/2 & b/2 \\ \bullet & b/2 & a/2 \end{array}$

Study <u>linear statistics</u> at large depth : $\sum_{i=1}^{k} c_i \cdot \#\{\text{color } i\}$.

Our case: $k = 2$, majority vote $\Longleftrightarrow$ $c_1 = +1$, $c_2 = -1$.

<u>Thm</u>: Root label $\underset{\uparrow}{=}$ majority at large depth $\Longleftrightarrow$ $\lambda_2(M)^2 > \lambda_1(M)$

Our case: $\underbrace{\boxed{\left(\frac{a-b}{2}\right)^2 > \frac{a+b}{2}}}$ (w.h.p.) [Mossel-Neeman-Sly '15]. generally related to eigenvector expansion etc.

Correct threshold for recoverability!

<u>Markov Chains</u>: General processes with "physics-like time": only present relevant in determining future.

<u>Countable case</u>: $(Y_n)_{n \geq 0}$ w/ $Y_n \in S$ countable (or finite) s.t.

$$\begin{aligned} &\mathbb{P}[Y_{n+1} = j \mid Y_n = i_n, \dots, Y_1 = i_1, Y_0 = i_0] \\ &= \mathbb{P}[Y_{n+1} = j \mid Y_n = i] =: p_n(i,j) \end{aligned}$$

<u>Markov property</u>

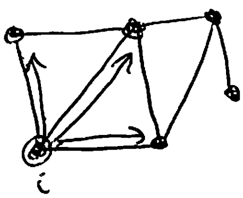
transition kernel.

$p_n(i,j) = p(i,j) \leftarrow$ <u>time-homogeneous</u> case. $p(i,j)$ = "prob. of going to j if now at i"

Ex: SRW $\cancel{Y}_n = \sum_{i=1}^n X_i \rightsquigarrow S = \mathbb{Z}$, $p(i,j) = \left\{\begin{array}{ll} 1/2 & \text{if } j = i+1 \\ 1/2 & \text{if } j = i-1 \\ 0 & \text{else} \end{array}\right\}$.

Ex: Branching process $Z_n \rightsquigarrow S = \mathbb{Z}_{\geq 0}$, $p(i,j) = \mathbb{P}\left[\sum_{a=1}^{i} X_{n,i} = j\right]$.

Ex: SRW on graph $\longrightarrow S = G$, $p(i,j) = \left\{\begin{array}{ll} 1/\deg(i) & \text{if } j \sim i \\ 0 & \text{else} \end{array}\right\}$.



Rk: Non-quantitative MC.

$(S, \mathcal{S})$ m'able space, $Y_n : \Omega \longrightarrow S$ r.v.'s,

General theory: (Y_n) adapted to $(\mathcal{F}_n)$. Would like to say:

$$\begin{array}{ccccc} \mathbb{P}[Y_{n+1} \in A \mid \mathcal{F}_n] & = & \mathbb{P}[Y_{n+1} \in A \mid Y_n] & \overset{(?)}{=} & p(Y_n, A) \\ \parallel & & \parallel & & \\ \mathbb{E}[\mathbb{1}\{Y_{n+1} \in A\} \mid \mathcal{F}_n] & & \mathbb{E}[\mathbb{1}\{Y_{n+1} \in A\} \mid Y_n] & & \end{array}$$

Def: $p : S \times \mathcal{S} \longrightarrow [0,1]$ is transition/Markov kernel if:

(1) $\forall A \in \mathcal{S}$, $y \longmapsto p(y, A)$ measurable function $\rightarrow$ (cf. previous discussion of conditioning on an r.v.)

(2) $\forall y \in S$, $A \longmapsto p(y, A)$ prob. measure on $(S, \mathcal{S})$.

Rk: Such regular conditional probability $\mathbb{P}[X \mid Y]$ does not always exist! But does for any remotely nice S (e.g., complete separable metric space w/ Borel σ-algebra.)

Def: (Y_n) is Markov chain w.r.t $(\mathcal{F}_n)$ and w/ kernel p if

$$\mathbb{P}[Y_{n+1} \in A \mid \mathcal{F}_n] = p(Y_n, A) \text{ a.s. } \forall n \geq 0,\ A \in \mathcal{S}.$$

Extend to all time by Kolmogorov ext. thm.

Given $Y_0 \sim \mu$ and p, can generate Markov chain: $p_y := p(y, \cdot) \rightsquigarrow$

$$\mathbb{P}[Y_0 \in A_0, Y_1 \in A_1, \ldots, Y_n \in A_n] := \int_{A_0} d\mu(y_0) \int_{A_1} dp_{y_0}(y_1) \cdots \int_{A_n} dp_{y_{n-1}}(y_n)$$