①

LECTURE 10: Recall "paradox" of mgl. betting strategy:

$$S_n = \sum_{i=1}^{n} X_i = \text{outcomes of bets}, \quad H_n = \text{bet sizes} = \begin{cases} 1 & n=1 \\ 2H_{n-1} & n>1,\ X_{n-1}=-1 \\ 0 & n>1,\ X_{n-1}=+1 \end{cases}$$

$(H \cdot S)_n$ = martingale of profit $\xrightarrow{a.s.} 1$.

$$H_n' := \begin{cases} 1 \\ 2H_{n-1} \\ 1 \end{cases} \longrightarrow \text{profit becomes arbitrarily large}$$

↳ restart instead of stopping.

Conceptual issue: finite available \$, need to account for running out.

I.e., if C = "bankroll", valid H will give $(H \cdot S)_n \geq -C$ always.

Thm: (Mgl. convergence) If (M_n) is $\left\{\begin{matrix} \text{mgl} \\ \text{submgl} \\ \text{supermgl} \end{matrix}\right\}$ and $\sup_n \mathbb{E}|M_n| < \infty$,

then $\exists$ r.v. M_∞ s.t. $M_n \xrightarrow{a.s.} M_\infty$ and $\mathbb{E}|M_\infty| < \infty$.

Cor: If (M_n) is supermgl. and $M_n \geq -C$ a.s. $\forall n$, then

$M_n \xrightarrow{a.s.} M_\infty$ and $\mathbb{E}M_\infty \leq \inf_n \mathbb{E}M_n \leq \mathbb{E}M_0$. ← "converges down."

Gambling consequences: with finite bankroll,

(1) Always keep playing $\Longrightarrow$ a.s. eventually run out of money.

(2) With any strategy, cannot profit on average.

Pf: (Cor. from Thm.) Supermgl $\Rightarrow \mathbb{E}M_n \leq \mathbb{E}M_0$

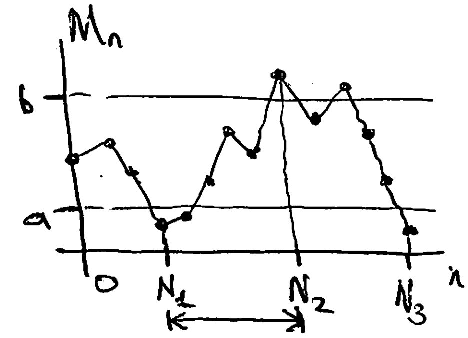
$\Longrightarrow \mathbb{E}|M_n| \leq \mathbb{E}[2C + M_n] \leq 2C + \mathbb{E}M_0$

Thm applies ⇝ limiting M_∞ exists.

$\mathbb{E}M_\infty \leq \inf_n \mathbb{E}M_n$ by Fatou Lemma (one-sided. bdd. conv.)

② Idea of pf of Thm : study interval crossings.

Def: Given (M_n) and $a<b$, define times:



$$N_0 := -1$$
$$N_{2k-1} := \min\{n > N_{2k-2} : M_n \leq a\}$$
$$N_{2k-2} := \min\{n > N_{2k-1} : M_n \geq b\}$$

Prop: $(N_i)_{i\geq 1}$ are all stopping times.

Def: The time period $[N_{2k-1}, N_{2k}]$ is an upcrossing of $[a,b]$.

$U_n = U_n(a,b) := \max\{k : N_{2k} \leq n\} = \#$ upcrossings by time n.

$U_\infty = U_\infty(a,b) := \lim_{n\uparrow\infty} U_n(a,b) = \#$ upcrossings in all time $\in \{0,1,2,\dots\}\cup\{+\infty\}$.

Lem1: If $\forall a,b\in\mathbb{R}$, $\mathbb{P}[U_\infty(a,b)<\infty]=1$, then $\mathbb{P}[M_n \text{ converges}]=1$.

Pf: Consider $a,b\in\mathbb{Q} \rightsquigarrow \mathbb{P}[U_\infty(a,b)<\infty\ \forall a,b\in\mathbb{Q}]=1$.

$$\mathbb{P}[M_n \text{ does not converge}] = \mathbb{P}[\liminf M_n \lneq \limsup M_n]$$
$$= \mathbb{P}[\exists a,b\in\mathbb{Q} : \liminf M_n < a < b < \limsup M_n]$$
$$\leq \mathbb{P}[\exists a,b\in\mathbb{Q} : U_\infty(a,b)=\infty] = 0.$$

Lem2: (Doob Upcrossing Lemma) $\mathbb{E}\,U_n(a,b) \leq \dfrac{\mathbb{E}(M_n-a)^-}{b-a}$ if M_n supermgl.

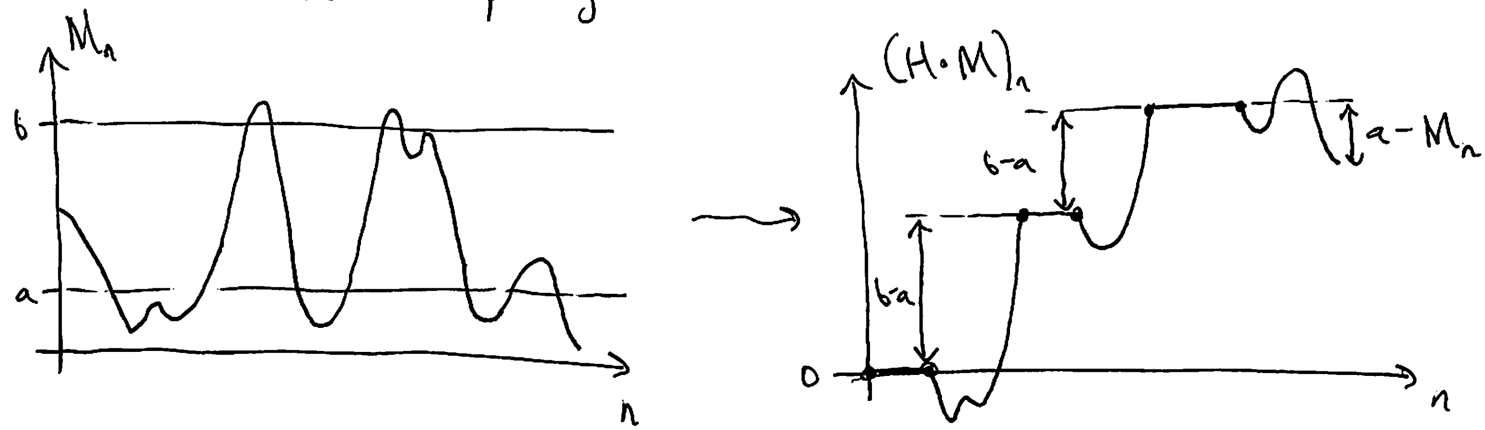
Def: $X^+ := X\vee 0$, $X^- := -(X\wedge 0)$ $\quad (X = X^+ - X^-,\ X^\pm \geq 0)$.

③ Pf: (Lem 2) "Buy low, sell high": $H_n := \mathbb{1}\{n \text{ is during upcrossing}\}$

$= \mathbb{1}\{N_{2k-1} \le n \le N_{2k}$ for some $k\} \ge 0$

N_i stopping times $\Rightarrow$ H_n predictable

$\Rightarrow$ $H \cdot M$ supermgℓ.



$$\Rightarrow (H\cdot M)_n \ge (b-a)\,U_n(a,b) - (a-M_n)^+$$

OTOH, $\mathbb{E}(H\cdot M)_n \le \mathbb{E}(H\cdot M)_0 = 0$. = "losing game".

Idea: Too many upcrossings would make betting on supermgℓ profitable.

Pf: (Thm) WLOG consider supermgℓ case. $\forall\, a<b$,

$$\mathbb{E}\,U_n(a,b) \overset{(\text{Lem } 2)}{\le} \frac{\mathbb{E}(a-M_n)^+}{b-a} \le \frac{|a| + \mathbb{E}|M_n|}{b-a} \overset{(\text{assumption})}{\le} C_{a,b}$$

(indep. of n)

$U_n(a,b) \uparrow U_\infty(a,b)$, monotone conv. $\Rightarrow$ $\mathbb{E}\,U_\infty(a,b) \le \underset{<\infty}{C_{a,b}}$

$\Rightarrow U_\infty(a,b) < \infty$ a.s.

Lem 1 $\Rightarrow$ $M_n \xrightarrow{a.s.} M_\infty$. Fatou Lemma $\Rightarrow$ $\mathbb{E}|M_\infty|$

$= \mathbb{E}\lim |M_n| < \infty$.

Rk: Possible to have $\mathbb{E}M_n \not\to \mathbb{E}M_\infty$; in Cor., can have strict $\mathbb{E}M_\infty \lneq \inf_n \mathbb{E}M_n$.