1

LECTURE 5

Tools for later:

Few more comments on cts. time:

① Kolmogorov ext. theorem

② Kolmogorov ~~continuity~~ theorem

Useful process model $f : \Omega \to (\mathbb{R}_{\geq 0} \to \mathbb{R})$ = law (prob. measure on $(\mathbb{R}^{\mathbb{R}_{\geq 0}}, B^{\mathbb{R}_{\geq 0}})$, equiv. to finite distributions) + restriction of path space $f(\omega) \in P'$ ALWAYS

Usually, P' not measurable in $B^{\mathbb{R}_{\geq 0}}$ (e.g. {cts functions}, {counting paths}

But no problem to still use restriction. Tricky!

Ex: Counting process is $f : \Omega \to (\mathbb{R}_{\geq 0} \to \mathbb{R})$ s.t. paths are always

$$f(\omega) \in P' = \left\{\begin{matrix} f(t) \in \mathbb{Z}_{\geq 0} \\ \text{non-decreasing} \\ \text{right-cts.} \end{matrix}\right\}$$

Note P' not measurable! (countable support argument)

PPP continued:

Thm: $\operatorname{BerP}(n, \lambda) \xrightarrow{(f.d.)} \operatorname{PPP}(\lambda)$. : Need fin. dist. of PPP.

Lem: (Law / finite distributions of PPP) If $f \sim \operatorname{PPP}(\lambda)$,

$$\operatorname{Law}\big((f(t_1), f(t_2) - f(t_1), \ldots, f(t_k) - f(t_{k-1}))\big)$$

product measure

$$= \operatorname{Pois}(\lambda t_1) \otimes \operatorname{Pois}(\lambda(t_2 - t_1)) \otimes \operatorname{Pois}(\lambda(t_k - t_{k-1}))$$

I.e.:

- # events in any interval Poisson
- Disjoint ~~event~~ intervals independent
- Interval distributions translation-invariant (stationary process)

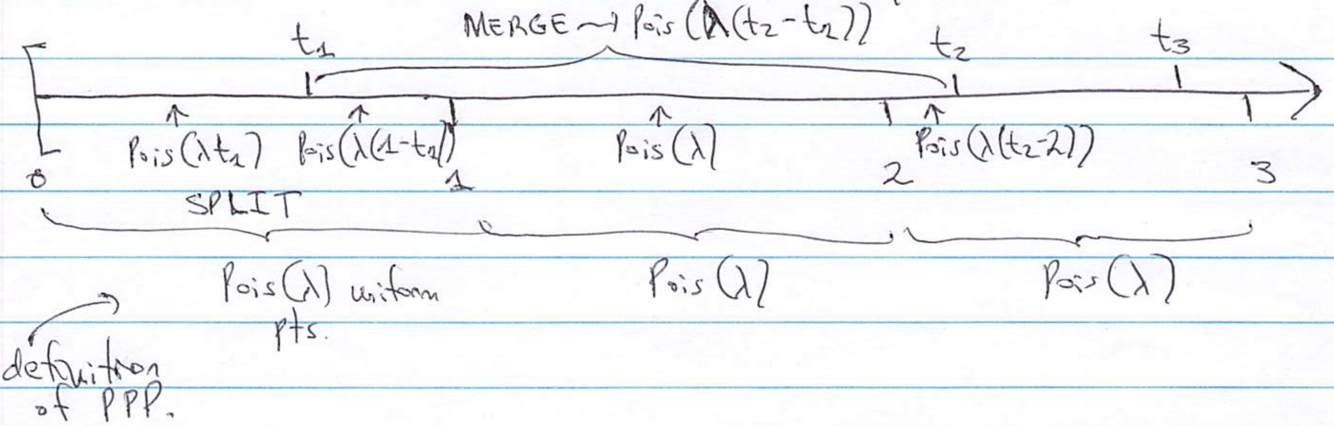
②

Facts: Important Poisson properties:

- **Merging:** $M \sim \mathrm{Pois}(\mu)$, $N \sim \mathrm{Pois}(\nu)$ ind. $\Rightarrow \mathrm{Law}(M+N) = \mathrm{Pois}(\mu+\nu)$.
- **Thinning:** $M \sim \mathrm{Pois}(\mu)$, $N \sim \mathrm{Ber}(M, p)$: split at random

$$\Longrightarrow \mathrm{Law}((N, M-N)) = \mathrm{Pois}(p\mu) \otimes \mathrm{Pois}((1-p)\mu).$$

Pf: (of Lem)



Rk: PLT says $f_n \sim \mathrm{BerP}(n, \lambda)$, $f \sim \mathrm{PPP}(\lambda)$ have $f_n(1) \Longrightarrow f(1)$

Pf: (of Thm) Using lemma, need $\mathcal{L}(f_n(t_1), \ldots, f_n(t_k) - f_n(t_{k-1})) \Rightarrow \mathrm{Pois}(\lambda t_1) \otimes \cdots \otimes \mathrm{Pois}(\lambda(t_k - t_{k-1}))$

Independence: by def. of BerP.

Convergence of coordinates: same as PLT.

Def: For f counting process, w/ $f(0) = 0$ a.s.,

$$T_k := \text{time of } k^{th} \text{ "jump"} = \inf\{t \geq 0 : f(t) \geq k\} \quad \text{(arrival time)}$$

$$T_0 := 0, \quad E_k := T_k - T_{k-1} \rightarrow \text{interarrival / waiting time.}$$

Rk: Only a random variable over process b/c. of path space restriction!

Thm: $f \sim \mathrm{PPP}(\lambda) \Longrightarrow \mathrm{Law}((E_1, \ldots, E_k)) = \mathrm{Exp}(\lambda) \otimes \cdots \otimes \mathrm{Exp}(\lambda)$

Def: $\mathrm{Exp}(\lambda)$ density $\lambda \exp(-\lambda x)\,dx$ on $x \geq 0$ **Prop:** $\mathbb{E}\, E_i = \frac{1}{\lambda}$.

Pf: Case of E_1: $\mathbb{P}[E_1 \le t] = \mathbb{P}[\underset{\substack{\uparrow \\ \mathrm{Pois}(\lambda t)}}{f(t)} = 0] = \frac{(\lambda t)^0}{0!} \underset{\substack{\uparrow \\ \text{cdf of } \mathrm{Exp}(\lambda)}}{\exp(-\lambda t)}$.

Cor: (Exp model of PPP) Draw $E_1, E_2, \ldots \overset{iid}{\sim} \mathrm{Exp}(\lambda)$.

Rk: Arrival times form random walk!

Set $T_k = \sum_{i=1}^k E_i$, $A := \{T_i\}$, f counting process assoc. to A.
Then, $\mathrm{Law}(f) = \mathrm{PPP}(\lambda)$. (i.e., fin dist. are same.)

Cor: $f_n \sim \mathrm{BerP}(n, \lambda) \longrightarrow (E_1(f_n), \ldots, E_k(f_n)) \Longrightarrow \mathrm{Exp}(\lambda)^{\otimes k}$.

Application: Extreme Value Theory (cf. HW 1).

$X_1, \ldots, X_n \sim \mathrm{Unif}([0,1])$. Sort: $0 \le X_{(1)} \le \cdots \le X_{(n)} \le 1$

Thm: If $f_n(t)$ jumps at $nX_{(1)}, \ldots, nX_{(n)}$, $f_n \xrightarrow{(f.d.)} \mathrm{PPP}(1)$.

Cor: $\left(nX_{(1)}, n(X_{(2)} - X_{(1)}), \ldots, n(X_{(k)} - X_{(k-1)})\right) \longrightarrow \mathrm{Exp}(1)^{\otimes k}$

Rk: $Y_i \overset{iid}{\sim} \mu$, $F_Y(t) := \mathbb{P}[Y_i \le t] \rightsquigarrow \mathrm{Law}(Y_i) = \mathrm{Law}(F_Y^{-1}(X_i))$
since $\mathbb{P}[F_Y^{-1}(X_i) \le t] = \mathbb{P}[X_i \le F_Y(t)] = F_Y(t) = \mathbb{P}[Y_i \le t]$.
$\rightsquigarrow$ description of extremes of general iid samples.

Will try to build tools to formalize statements like this:

Thm: (Watanabe '64) $f \sim \mathrm{PPP}(\lambda)$ is the only pt. process s.t.:
- $\mathbb{E} f(t) = \lambda t$
- $f(t) - \lambda t$ is an "unpredictable" process. (martingale)

Formalize: "future independent of what has happened so far."