# Assignment 3

Probability Theory II
(EN.553.721, Spring 2025)

Assigned: March 6, 2025     Due: 11:59pm EST, March 17, 2025

**Solve any three out of the four problems.** If you solve more, we will grade the first three solutions you include. Each problem is worth an equal amount towards your grade.

Submit solutions in LaTeX. Write in complete sentences. Include and justify all steps of your arguments, but avoid writing excessive explanation that is not contributing to your solution.

Keep in mind the late submission policy: you may use a total of five late days for homework submissions over the course of the semester without penalty. If you need an extension beyond these, *you must ask me 48 hours before the due date of the homework* and have an excellent reason. After you have used up these late days, further late assignments will be penalized by 20% per day they are late.

For some of these problems, you might find it useful to watch the recorded lectures on martingales if you haven't yet.

**Problem 1** (Martingales for random walks)**.** Let $X_i \sim \mathrm{Unif}(\{\pm 1\})$ be i.i.d., and let $S_n = \sum_{i=1}^n X_i$ with $S_0 = 0$ be the usual simple random walk.

1. Let $T := \min\{n : S_n = a\}$ be the hitting time of some $a > 0$. Using the optional stopping theorem, show that $\mathbb{E}T = \infty$.

2. Using Doob's martingale convergence theorem, show that $T < \infty$ almost surely.

   (Hint: Rephrase this conclusion as the convergence almost surely of a suitable martingale.)

3. Let $\lambda \in \mathbb{R}$ and
$$M_n := \frac{\exp(\lambda S_n)}{\cosh(\lambda)^n}.$$
   Show that $M_n$ is a martingale.

4. Using the optional stopping theorem, derive a formula for $f(z) = \mathbb{E}z^T$ valid for all $0 \le z \le 1$. Note that, by our recent discussion in class, the Taylor series expansion of this function contains the values of $\mathbb{P}[T = k]$ for all $k \ge 0$. (You are not required to derive these.)

**Problem 2** (Time to observe a sequence)**.** Let $a_1, \dots, a_N \in \{0,1\}$ be fixed, and let $X_1, X_2, \dots \sim \text{Unif}(\{0,1\})$ be i.i.d. random variables. In this problem, you will study the time required to see the sequence $(a_1, \dots, a_N)$ in the random sequence $(X_1, X_2, \dots)$:

$$T := \min\{n : n \geq N, X_n = a_N, X_{n-1} = a_{N-1}, \dots, X_{n-N+1} = a_1\}.$$

Consider a Casino where a countable number of Gamblers $1, 2, \dots$ bet on the outcome of the $X_n$ at each time $n$. Gambler $i$ bets $B_{i,n,0}$ on the outcome being $X_n = 0$ at time $n$, and $B_{i,n,1}$ on the outcome being $X_n = 1$ at time $n$. At each time $n$, almost surely only a finite number of the Gamblers make a non-zero bet. If $X_n = 0$, then the Casino's fortune increases by $B_{i,n,1}$ for each $i$ and decreases by $B_{i,n,0}$ for each $i$, and Gambler $i$'s fortune increases by $B_{i,n,0}$ and decreases by $B_{i,n,1}$; if $X_n = 1$, the same happens with 0 and 1 switched. Write $(G_{i,n})_{n\geq 0}$ for the fortune of Gambler $i$ at each time.

1. Suppose that the processes $(B_{i,n,s})_n$ for each $i \geq 1$, $s \in \{0,1\}$ are predictable with respect to the $\sigma$-algebra $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Define the net profit of the Casino from the bets made at time $n$ as:

$$Y_0 := 0,$$
$$Y_n := \sum_{i\geq 1} \mathbb{1}\{X_n = 0\}(B_{i,n,1} - B_{i,n,0}) + \sum_{i \geq 1} \mathbb{1}\{X_n = 1\}(B_{i,n,0} - B_{i,n,1}).$$

   Define the net profit up to time $n$ as:

$$M_n := \sum_{j=1}^{n} Y_j.$$

   Show that $M_n$ is a martingale.

2. The Gamblers play strategies similar to but slightly different from the martingale betting strategy from class. Each Gambler starts with a fortune of $G_{i,0} = 1$. Gambler 1 bets all of their money on $X_i = a_i$, until either losing and stopping or leaving with a fortune of $2^N$:

$$B_{1,n,s} := \mathbb{1}\{1 \leq n \leq N, X_1 = a_1, \dots, X_{n-1} = a_{n-1}, s = a_n\} \cdot 2^{n-1}.$$

   The other Gamblers $j \geq 2$ play the same strategy, but starting at time $j$:

$$B_{j,n,s} := \mathbb{1}\{j \leq n \leq N + j - 1, X_j = a_1, \dots, X_{j+n-2} = a_{n-1}, s = a_n\} \cdot 2^{n-j}.$$

   Suppose that $N = 2$ and $(a_1, a_2) = (0, 1)$. Give a formula for the value of $M_T$ in terms of $T$ for each $T \geq 2$. Compute $\mathbb{E}[M_T]$ using the optional stopping theorem. Use that to compute $\mathbb{E}[T]$.

3. Repeat Part 2 for $N = 6$ and $(a_1, a_2, a_3, a_4, a_5, a_6) = (1, 1, 0, 0, 1, 1)$.

4. Describe a general formula for $\mathbb{E}[T]$ in terms of $(a_1, \dots, a_N)$. For a given length $N$, find a sequence that makes $\mathbb{E}[T]$ as large as possible and another that makes $\mathbb{E}[T]$ as small as possible.

**Problem 3** (Product martingales)**.** Let $X_1, X_2, \ldots$ be independent random variables such that $X_i \geq 0$ almost surely and $\mathbb{E}X_i = 1$ for all $i$. We have seen that

$$M_0 := 1,$$
$$M_n := \prod_{i=1}^{n} X_i \text{ for } n \geq 1$$

defines a martingale. Define $s_i := \mathbb{E}\sqrt{X_i}$.

1. Show that $0 < s_i \leq 1$ for all $i$.
2. Show that there exists a random variable $M_\infty \geq 0$ such that $M_n \to M_\infty$ almost surely.
3. Show that if $\prod_{i=1}^{\infty} s_i = 0$, then $M_\infty = 0$ almost surely.

   (**Hint:** Consider another martingale formed by suitably normalizing $\prod_{i=1}^{n} \sqrt{X_i}$.)
4. Show that if $\prod_{i=1}^{\infty} s_i > 0$, then $M_n \to M_\infty$ in $L^1$, and therefore $\mathbb{E}M_\infty = 1$ and it is not the case that $M_\infty = 0$ almost surely.

   (**Hint:** Use the same martingale from Part 3 and a suitable martingale maximal inequality to establish that $\sup_n M_n < \infty$ almost surely. Then, apply dominated convergence.)

**Problem 4** (Sums of random length)**.** Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{E}|X_i| < \infty$ and let $S_n := \sum_{i=1}^{n} X_i$. Let $N \in \mathbb{N}$ be a random variable independent of the $X_i$ having $\mathbb{E}N < \infty$.

1. Show that $\mathbb{E}S_N = \mathbb{E}\sum_{i=1}^{N} X_i = \mathbb{E}N \cdot \mathbb{E}X_1$.

   (**Hint:** Write $S_N = \sum_{i=1}^{\infty} X_i \mathbb{1}\{i \leq N\}$. Handle the case $X_i \geq 0$ first.)
2. Show that if further $X_i \geq 0$ almost surely, $X_i$ is not almost surely zero, and $\mathbb{E}N = \infty$, then the above identity still holds, in the sense that $\mathbb{E}S_N = \infty$.
3. Show that, if $\mathbb{E}X_i^2 < \infty$, then $\mathbb{E}(S_N - N\mathbb{E}X_1)^2 = \mathbb{E}N \cdot \mathrm{Var}[X_1]$ and conclude that, if also $\mathbb{E}N^2 < \infty$, then $\mathbb{E}S_N^2 = \mathbb{E}N \cdot \mathbb{E}X_1^2 + \mathbb{E}[N(N-1)] \cdot (\mathbb{E}X_1)^2$.

   (**Hint:** Reduce to the case $\mathbb{E}X_1 = 0$. Use the $L^2$ martingale convergence theorem.)
4. Calculate the mean and variance of the sum of the rolls of a six-sided die made until a 6 is rolled.